Project: When was the CMB generated?

The purpose of this project is to apply the thermal physics to the evolution of the universe to obtain the temperature $T_{\rm dec}$[1] when the cosmic microwave background (CMB) was generated in the early universe. Here we will choose a simple approach to find $T_{\rm dec}$. Adopting the natural unit $c = \hbar = k_B = 1$ turns out useful. Follow the steps below to find your result!

1. **Number densities of particles** Assume for simplicity (and in fact it's a good approximation) that there are photons γ, electrons e and protons p. The relevant equilibrium process with these particle contents before the CMB is $p + e \leftrightarrow {}^1H + \gamma$, with ${}^1H$ being a neutral hydrogen atom. Use the phase space distribution functions in kinetic equilibrium of each particle $f_i(\boldsymbol{p})$ ($i = \gamma$, e and p) to find the number density n_i. Bear in mind the following:

   - Of course, electrons and protons are fermions while photons are bosons. Use different distribution functions for them.
   - Since photons are involved, the energy should be regarded as relativistic, i.e. $E = \sqrt{m^2 + p^2}$ where $p = |\boldsymbol{p}|$.
   - For electrons and protons, assume that the rest masses are much larger than temperature.

2. **Saha equation** Now we proceed to write down the relation between the temperature and the "ionization fraction", $X_e \equiv n_e/(n_e + n_{^1H}) = n_p/(n_p + n_{^1H})$ where the 2nd equality follows from the neutrality of the universe, $n_e = n_p$. Combining the number densities n_e, n_p and $n_{^1H}$, we can eliminate the chemical potentials. Denoting $B \equiv m_e + m_p - m_{^1H} \approx 13.6$ eV being the binding energy of hydrogen, find the following relation:

$$\frac{1 - X_e}{X_e^2} \approx \eta \frac{4\sqrt{2}\zeta(3)}{\sqrt{\pi}} \left(\frac{T}{m_e}\right)^{3/2} e^{B/T} \, .$$

   This equation is called the Saha equation. Here, $\eta \equiv (n_p + n_{^1H})/n_\gamma$ is called the baryon-to-photon ratio[2] and $\zeta(s)$ is the Riemann zeta function. Remember:

   - We assume the universe is in thermal equilibrium. This means for the whole process $p + e \leftrightarrow H + \gamma$ the chemical potential vanishes.
   - Properly take into account the number of spin states g_i. For example, $g_{^1H} = 4$. Can you guess why?

3. **Decoupling temperature** Given the Saha equation, we can see how X_e behaves as a function of the temperature. How does X_e change as T becomes smaller? If we define recombination as the point when 90% of the electrons have combined with protons, what is the corresponding decoupling temperature $T_{\rm dec}$ with $\eta \approx 6.2 \times 10^{-10}$? Do you think the value of $T_{\rm dec}$ you find is natural? How old is the universe at that time?

   - For the last question, use the Friedmann equation in the matter dominated era.

[1]The subscript "dec" stands for decoupling since, as will be explained during the lecture, the CMB was generated when the photons and the free electrons were decoupled.

[2]In fact, about 25% of protons exist in the form of helium nuclei.